NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: BACHELOR OF SCIENCE; BACHELOR OF SCIENCE IN APPLIED MATHEMATICS AND STATISTICS	
QUALIFICATION CODE: 07BSOC; 07BAMS	**LEVEL:** 6
COURSE CODE: LIA601S	**COURSE NAME:** LINEAR ALGEBRA 2
SESSION: NOVEMBER 2019	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER:	MR G. TAPEDZESA
MODERATOR:	MR B. OBABUEKI

INSTRUCTIONS
1. Examination conditions apply at all times. **NO** books, notes, or phones are allowed.
2. Answer **ALL** the questions and number your answers clearly and correctly.
3. Marks will not be awarded for answers obtained without showing the necessary steps leading to them (the answers).
4. Write clearly and neatly.
5. All written work must be done in dark blue or black ink.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

QUESTION 1. [27 MARKS]

1.1 Let $T : P_2 \to P_2$ be a mapping defined by

$$T(a_0 + a_1x + a_2x^2) = 5a_0 + a_1x^2, \quad \text{where } a_0, a_1, a_2 \in \mathbb{R}$$

(a) Show that T is a linear mapping. [5]

(b) Hence, find the kernel of T. [5]

(c) Is T one-to-one? Explain your answer. [2]

1.2 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator for which $T(1,1) = (1,-2)$ and $T(1,0) = (-4,1)$. By noting that $\{(1,1),(1,0)\}$ is a basis of $\mathbb{R}^2$, find a formula for $T(x,y)$, and then use the formula to compute $T(5,-3)$. [7]

1.3 What does it mean to say that a linear mapping $T : V \to W$ is singular? [2]

1.4 Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mappings defined by

$$T_1(x,y,z) = (z-x, 2y), \quad T_2(x,y) = (x+y, x-y), \quad \text{and} \quad T_3(x,y) = (y, 0, 2x).$$

Find a formula for defining each of the following compositions, if possible. If it is not possible to have such a formula, give a reason.

(a) $T_2 \circ T_1$ [3]

(b) $T_3 \circ T_2$. [3]

QUESTION 2. [28 MARKS]

2.1 Consider the linear operator T on P_2, defined by

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(2x+1) + a_2(2x+1)^2,$$

and the basis $S = \{1, x, x^2\}$ in P_2.

(a) Find the matrix representation of T relative to S. [7]

(b) By observing that S is the standard basis for P_2, or otherwise, find the coordinate vector for $p = 2 - 3x + 4x^2$ relative to the basis S, and denote it by $[p]_S$. [2]

(c) Use the transition matrix you obtained in part (a) above and the result in (b) to compute $[T(p)]_S$. [4]

(d) Hence, determine $T(p) = T(2 - 3x + 4x^2)$. [2]

2.2 Consider the following two bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ for $\mathbb{R}^2$, where

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

(a) Find the transition matrix from B to B' and denote it by $P_{B \to B'}$. [7]

(b) Compute the coordinate vector $[w]_B$ where $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and, hence, use the transition matrix you obtained in part (a) above to compute $[w]_{B'}$. [6]

QUESTION 3. [22 MARKS]

3.1 Suppose that the characteristic polynomial of some square matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

(a) What is the size of the matrix A? [2]

(b) Is the matrix A invertible? [2]

(c) How many eigenspaces does A have? [2]

Explain your answers.

3.2 Suppose $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Confirm that P diagonalises A, by finding P^{-1} and directly computing $P^{-1}AP$. [9]

(b) Hence, find A^{1000}. [7]

QUESTION 4. [23 MARKS]

4.1 Let V be a finite dimensional vector space over a field K.

(a) What does it mean to say that a mapping $f : V \times V \to K$ is a bilinear form on V? [3]

(b) What does it mean to say that a bilinear form f, as defined in (i) above, is symmetric? [3]

(c) What does it mean to say that a mapping $Q : V \to K$ is a quadratic form on V? [3]

QUESTION 4 CONTINUES ON THE NEXT PAGE

4.2 Consider the equation $5x_1^2 - 4x_1x_2 + 8x_2^2 = 36$.

(a) Re-write the equation in the matrix form $\mathbf{x}^T A\mathbf{x} = 36$, where A is a symmetric matrix. [4]

(b) Given that the matrix

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

orthogonally diagonalises A, use a suitable variable transformation to place the conic in standard position and, hence, identify the conic section represented by the equation. [10]

END OF QUESTION PAPER